Sand Boils and Liquefaction-Induced Lateral Deformation

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ABSTRACT: A mechanism of sand boil development during liquefaction is briefly discussed. In particular, the impact of sand boils as potential excess pore-pressure relief mechanisms is investigated within a computational solid-fluid fully coupled effective-stress framework. In this regard, sand boils may contribute to significant reduction in liquefaction-induced lateral ground deformations (in profiles vulnerable to flow failure scenarios). Illustrations of representative sand boil features are discussed, along with related liquefaction laboratory test results. Thereafter, a computational investigation is presented to highlight: 1) impact of overall site permeability in dictating the nature and magnitude of liquefaction-induced lateral deformation, and 2) potential effect of sand boils on the magnitude of such deformations.

1 INTRODUCTION

Liquefaction-induced lateral deformation continues to be a major cause of earthquake-related damage (Seed et al. 1990, Ishihara et al. 1990, Bardet et al. 1995, Sitar 1995, Soils and Foundations 1996, see also http://geoinfo.usc.edu). In liquefaction susceptibility/triggering analyses (Seed and Idriss 1982, Kramer 1996, Youd and Idriss 1997), the assumption of undrained excess pore-pressure (u_e) buildup is usually employed. However, the presence of sand boils, as a potential excess pore-pressure relief mechanism, may significantly modify the u_e buildup/dissipation pattern in a soil profile. Thus, sand boils may have potentially a major influence on post-liquefaction soil behavior, particularly in limiting liquefaction-induced lateral deformations.

In this paper, the mechanisms of sand boil generation and evolution will be briefly discussed, based on evidence from field observations and earlier experimental/theoretical studies. Generally, it will be shown (as reported in earlier studies) that inhomogeneities within the soil mass (in soil-type, permeability) may promote the development of sand boils.

A pilot numerical study is also presented to illustrate the impact of sand boils on liquefaction-induced lateral deformations in stratified soil profiles. This numerical study was performed using a solid-fluid fully coupled Finite Element program (CYCLIC, Parra 1996, Elgamal et al. 1999, Yang 2000) that incorporates a newly developed liquefaction constitutive model (Parra 1996, Yang 2000). The discussion below may be considered as an initial attempt to explore the relevance of sand boils to practical engineering applications.

2 FIELD OBSERVATION AND PHYSICAL MECHANISM

Sand boils are observed during/after major earthqaukes (e.g., Charleston 1888, San Francisco 1906, Alaska and Niigata, 1964, Imperial Valley 1979, Loma Prieta, 1989, etc.). These features (Figs. 1 and 2) may develop during or after a strong ground shaking phase and appear in various sizes (in a conical shape or along a longitudinal crack). The ejected water-soil mixture has been reported to reach heights in excess of 4 ft (1.2 m) above the ground (e.g., Kawakami 1965). The amount of fluid extruded to the surface depends on the degree of "damage" inflicted on the solid particle skeleton. For very loose soil formations, 15% or more of the pore fluid may be extruded upwards due to cyclic loading.

Fig. 1. California [highway] 98, 1 mi. east of Anderholt Road. Imperial Valley, California, earthquake Oct. 15, 1979 (Steinbrugge Collection, Earthquake Engineering Research Center, University of California, Berkeley, http://nisee.berkeley.edu).

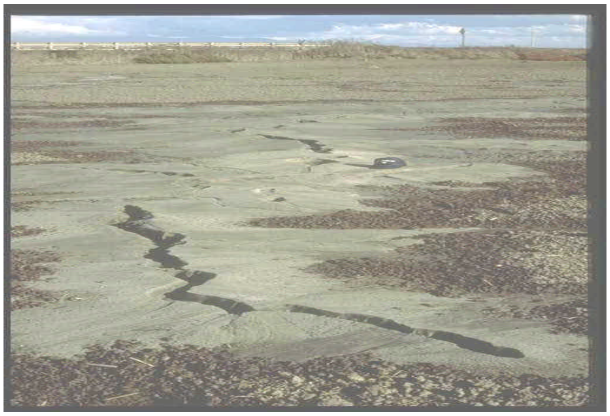

Fig. 2. Liquefaction of sandy soils in abandoned channel of Salinas River. Loma Prieta, California earthquake, Oct. 17, 1989 (Loma Prieta Collection, Earthquake Engineering Research Center, University of California, Berkeley, http://nisee.berkeley.edu).

Various theoretical and experimental attempts have been made in the past to explain the mechanisms of sand boils. Theoretical work of Housner (1958), Ambraseys and Sarma (1969), and experimental studies by Scott and Zuckerman (1972), Liu and Qiao (1984), Elgamal et al. (1989), and Kokusho (1999) attributed the formation of sand boils to inhomogeneity in soil properties (especially spatial variation in permeability) near the ground surface. Waller (1966) and Segerstorm et al. (1963) mention, "These extrusions indicate that the high pressures must have existed in the water-table aquifer and thus a temporary confining layer must have been present." Scott and Zuckerman (1972), in an extensive insightful discussion of the sand boil formulation mechanisms, concluded that the presence of silt or a suitably fine-grained layer at the surface is conducive to the generation of sand boils.

Many natural and man-made liquefiable sand deposits contain finer, more impervious silty or clayey layers (a typical example is soil strata generated by the hydraulic fill process, Seed 1987). Inhomogeneity in soil properties can lead to a difference in the rate of fluid extrusion and hence create a chance for fluid migration towards more pervious and/or lower pressure locations. During the migration process, the fluid may be entrapped underneath a stratum of relatively lower permeability, and forms a water-rich layer thereby (Fig. 3). A laboratory test conducted by Elgamal et al. (1989) of an interlayered sand-clay-sand soil profile shows: a) the relatively large thickness of accumulated water interlayers which may develop, and b) the long time during which this thickness is sustained following the end of dynamic load.

At an inhomogeneity (e.g., crack, void, etc.) a fingering process may start within the soil mass and make its way to the surface in the form of a soil boil (Fig. 3). The size of this boil will depend in general on the available fluid to be ejected during its formation. Scott and Zuckerman (1972), Liu and Qiao (1984), and Elgamal et al. (1989) observed sand boils in small-scale tests of stratified deposits (of varying permeability), following the water interlayer formation along the boundary of lower coarser and upper finer layers. It may be roughly concluded that boils that eject large amounts of saturated soil occur mainly during and soon after an earthquake. At this stage, dispersed soil may be available for ejection. Once the soil sediments, large craters may still develop if the released fluid is trapped below an impervious boundary. In this case, most of the ejected soil will be that of overlying soil along with large amounts of almost pure fluid (Fig. 3).

Kutter and Fiegel (1991) describe the development of a sand boil in a centrifuge experiment in which an upper low permeability silt layer overlies loose sand. They reported that there is some evidence that a water gap or a very loose zone of soil developed at the interface between the sand and silt layers. Locations of sand boils appeared to be concentrated near zones of weakness in the silt layer, and the boils were caused by venting of water through the developed gaps.

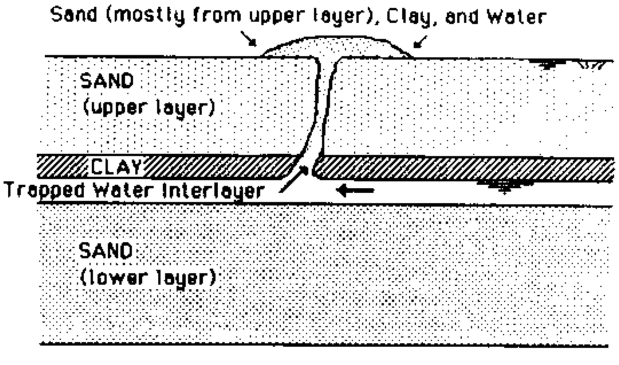

Fig. 3. Formation of trapped water interlayer, and delayed sand boil following a hydraulic fracture-mechanism (Elgamal et al. 1989).

3 SAND BOIL AND LIQUEFACTION-INDUCED LATERAL DEFORMATION

In the case of a sloping liquefiable soil profile, liquefaction-induced water interlayers underneath lower-permeability strata are potentially of primary significance in dictating liquefaction-associated lateral deformation (Kokusho 1999). The shear strength along a water interlayer often approaches zero. Such an excessively weak interface may cause large lateral ground deformation (and lateral loading on underground structures) even in mildly sloping terrain. Since the water interlayers might not dissipate for hours or even days after an earthquake, lateral deformations may accumulate long after the end of the shaking. However, lateral deformation and associated tension cracking will in general create sand boils allowing any interlayer water to escape, and eventually bringing the deformation process to an end.

Recently, Kokusho (1999) conducted insightful 1D and 2D liquefaction shaking table tests to demonstrate the evolution of a water film (trapped below a silt seam), and its key role in dictating the time of occurrence and extent of lateral deformation in sloping ground. In a 2D embankment test, the portion of the embankment above the water interlayer started to slide several seconds after the shaking event, resulting in a (delayed) flow failure. Earlier, a number of centrifuge model tests were also conducted to investigate the effect of permeability variation on lateral spreading, including Arulanandan et al. (1988), Arulanandan and Scott (1993, 1994), Kutter and Fiegel (1991), and Balakrishnan and Kutter (1999). Most of these experiments employed clean liquefiable sand profiles overlain by a clay/silt layer.

4 NUMERICAL STUDY

Numerical simulations are conducted herein to study the potential effects of sand boils on liquefaction-induced lateral deformation. These numerical simulations are performed using the solid-fluid fully coupled Finite Element program CYCLIC (Parra 1996, Elgamal et al. 1999, Yang 2000) that incorporates a newly developed liquefaction constitutive model (Parra 1996, Yang 2000). In the following sections, the employed computational formulation and soil model are briefly described. Numerical simulation results are then presented and discussed in detail.

4.1 *Computational formulation*

In CYCLIC, soil is modeled as a two-phase material based on the Biot (1962) theory for porous media. A simplified numerical formulation of this theory, known as *u-p* formulation (in which displacement of the soil skeleton *u*, and pore pressure *p*, are the primary unknowns, Zienkiewicz et al. 1990), was implemented in CYCLIC (Ragheb 1994, Parra 1996, Yang 2000). The computational scheme follows the methodology of Chan (1988).

The incorporated liquefaction model is based on the original multi-yield-surface plasticity concept (Prevost 1985), In this model, much modification was made with emphasis placed on controlling the magnitude of liquefaction-induced cycle-by-cycle permanent shear strain accumulation in clean medium-dense sands (Parra 1996, Yang 2000). The resulting newly developed constitutive model has been extensively calibrated for clean Nevada Sand at a relative density of about 40% (Parra 1996, Elgamal et al. 1999, Yang 2000). The calibration phase included results of laboratory sample tests (Arulmoli et al. 1992) as well as data from dynamic centrifuge-model simulations (Dobry et al. 1995, Taboada 1995). In the following numerical simulations, this calibrated model is employed to represent the dynamic properties of sand material (with a different permeability coefficient).

4.2 *Computational simulation model*

Three two-dimensional simulations were conducted, all employing a rectangular Finite Element mesh of 30m in width and 11m in height, inclined at a mild slope of 4 degrees (Fig. 4). The water table was set at the ground surface. For all cases, the lower 9m consisted of a liquefiable medium sand material (permeability coefficient = 6.6x10^{-4} m/s). In Case 1, a 2m clay layer with a very low permeability coefficient completely covered the sand stratum; In Cases 2 and 3, the overlying clay layer included 1 and 3 sand elements, respectively (Fig. 4). In the two-dimensional context, the presence of these sand elements mimics the high-permeability surface fissures extending in the out-of-plane direction (similar to Fig. 2). Finally, a very thin layer of sand elements was placed under the clay cap (Fig. 4), for simulating the possible localized accumulation of lateral deformations at this elevation, as described earlier.

The employed boundary conditions for all cases are:

- For the fluid phase, the base and lateral boundaries are impervious, with zero prescribed pore pressure at the surface.
- For the solid phase, dynamic excitation (Fig. 5) was about 15 seconds of mainly 2 Hz harmonic base acceleration. Nodes at the same elevation along the two lateral boundaries were tied horizontally (using a penalty method) to mimic 1D overall shear response (Parra 1996).

A static application of gravity was performed before seismic excitation (applying the soil own weight). The resulting fluid hydrostatic pressures and soil stress-states served as initial conditions for the subsequent dynamic analysis.

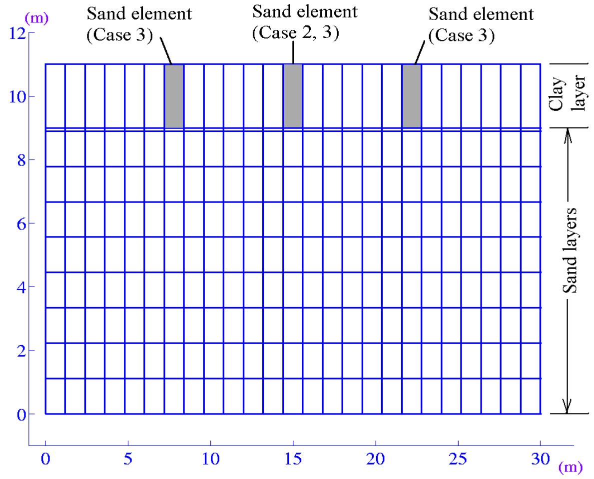

Fig. 4. Finite element mesh for numerical simulations (inclined 4 degrees to horizontal).

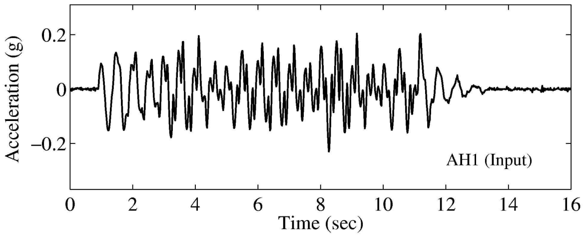

Fig. 5. Employed input base acceleration record.

5 RESULTS AND DISCUSSIONS

5.1 *Case 1: No sand boil*

Fig. 6 depicts u_e time histories along the vertical centerline of the mesh. During the shaking phase, u_e developed within the sand layer, with higher values at lower elevations (Fig. 7). After the shaking phase, pore pressure rapidly dissipated in the lower portions of the sand deposits, eventually reaching a steady value of about 18 kPa (Figs. 6 and 8). This steady value is equal to the initial effective overburden pressure of the overlying clay cap, i.e., the thin sand layer underneath the clay cap lost its effective confinement completely. Due to the very low permeability of the clay cap prohibiting rapid upward dissipation of u_e, this sand layer became essentially a water interface (see Section 2), and remained liquefied long after the earthquake (Figs. 6 and 8). The shear strength of this thin layer was even lower than the static driving shear stress (due to the 4 degrees inclined self-weight component of the overlying clay layer), resulting in very large post-liquefaction lateral sliding as shown in Fig. 9. Note that this continuous lateral sliding only occurred at the top of the thin layer, translating the clay cap in the down-slope direction. Beneath this thin layer, however, lateral deformation stopped immediately after the shaking phase (Fig. 9). This pattern of large lateral deformation concentration is clearly exhibited in Fig. 10.

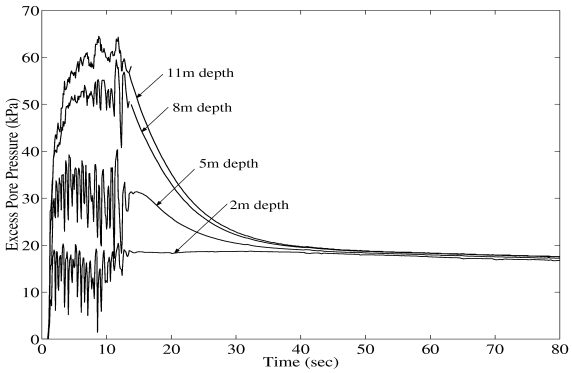

Fig. 6. Excess pore pressure histories along the centerline of the mesh (no sand boil case).

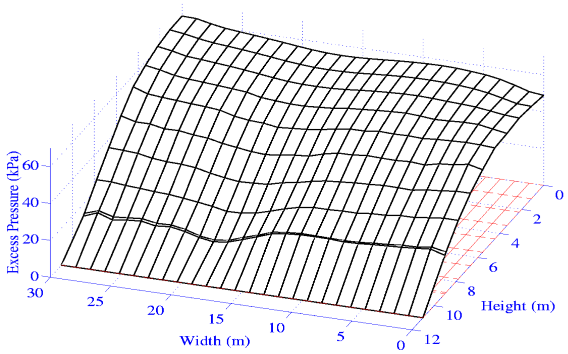

Fig. 7. Excess pore pressure profile at the end of shaking (no sand boil case).

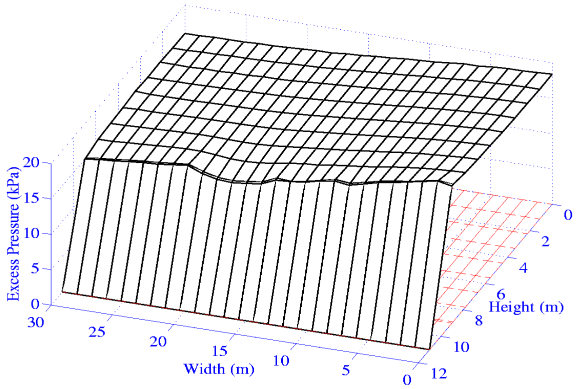

Fig. 8. Excess pore pressure profile at about 1 minute after the shaking phase (no sand boil case).

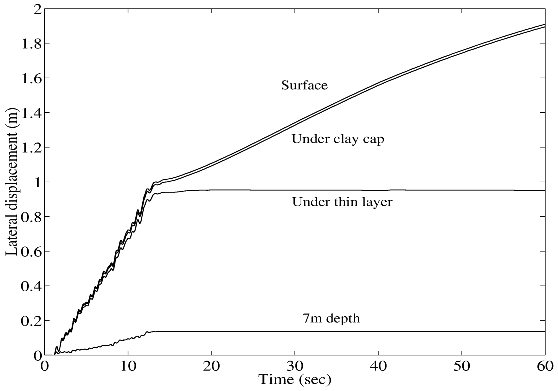

Fig. 9. Lateral displacement time histories along the mesh centerline (no sand boil case).

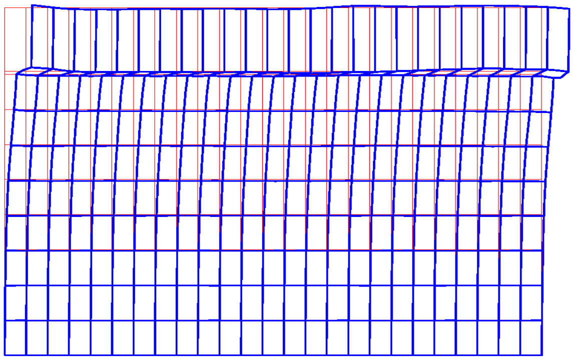

Fig. 10. Deformed mesh at about 1 minute after the shaking phase (no sand boil case).

5.2 *Case 2: one sand boil simulation*

In this case, u_e buildup during the shaking phase is nearly identical to Case 1 (Fig. 11), followed by a rapid upward dissipation phase. Because excess pore fluid may escape through the sand element in the middle of the clay layer, it is observed from Figs. 11 and 12 that: 1) the overall u_e profile is lower than that of Case 1 one minute after the shaking phase, and 2) a significant pore pressure variation in the horizontal direction, with the lowest values along the mesh centerline (closet to the simulated sand boil location). Lateral displacement histories along the centerline (Fig. 13) show a short period of relatively mild post-shaking sliding, which stopped quickly as the sand layer underneath the clay cap regained effective confinement. The overall deformation pattern in this case (Fig. 14) is fairly smooth and uniform.

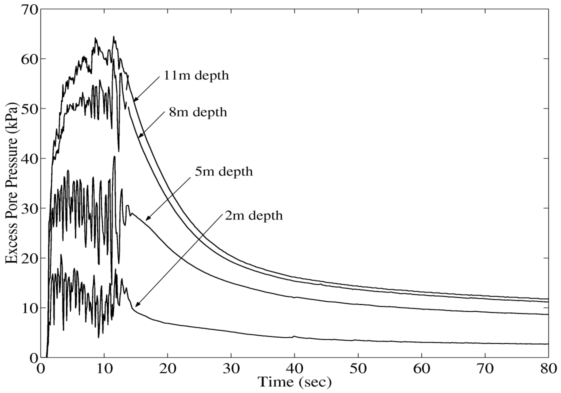

Fig. 11. Excess pore pressure histories along the mesh centerline (1 sand boil simulation).

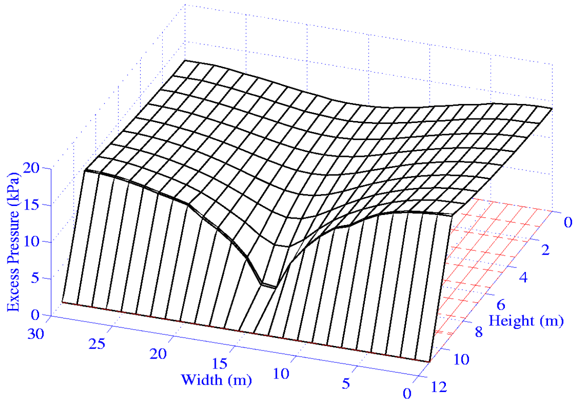

Fig. 12. Excess pore pressure profile at about 1 minute after the shaking phase (1 sand boil simulation).

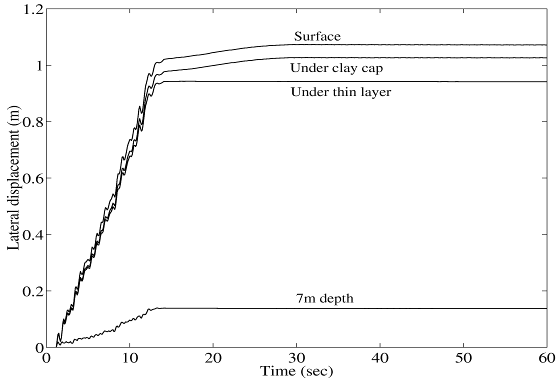

Fig. 13. Lateral displacement time histories along the mesh centerline (1 sand boil simulation).

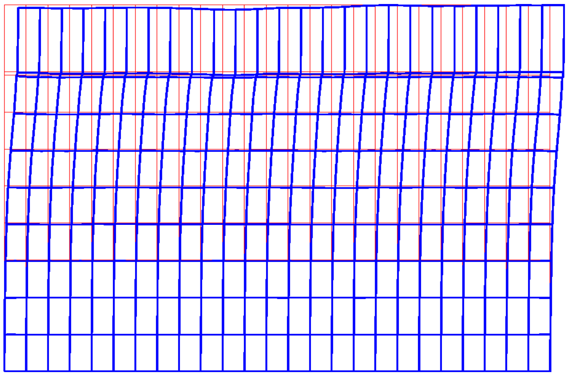

Fig. 14. Deformed mesh at about 1 minute after the shaking phase (1 sand boil simulation).

5.3 *Case 3: three sand boil simulation*

With 3 sand elements present in the clay layer (Fig. 4), u_e dissipated even more rapidly after the shaking phase (Fig. 15), resulting in a u_e profile much lower than the first two cases (Fig. 16). In addition, no post-shaking sliding is observed in the entire mesh (Fig. 17). The overall deformation pattern in this case is essentially similar to Case 2.

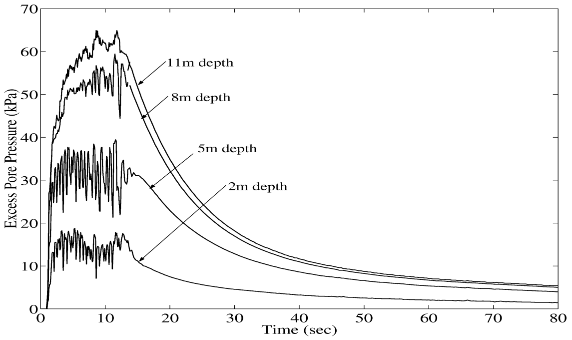

Fig. 15. Excess pore pressure histories along the centerline of the mesh (3 sand boil simulation).

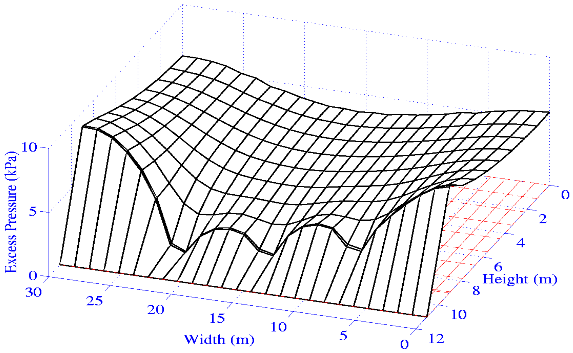

Fig. 16. Excess pore pressure profile at about 1 minute after the shaking phase (3 sand boil simulation).

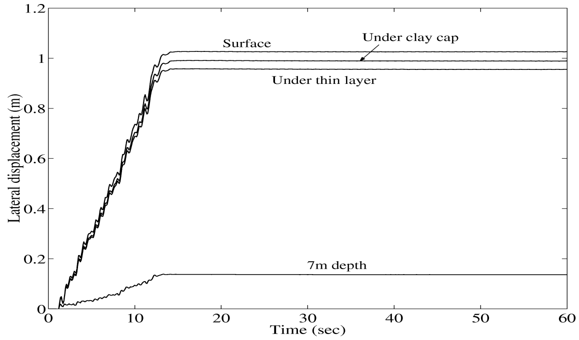

Fig. 17. Lateral displacement time histories along the mesh centerline (3 sand boil simulation).

5.4 *Discussion*

1. In all three cases, it is seen that soil response (pore pressure and lateral displacement) during the shaking phase is practically unaffected by presence of the simulated sand boils. This is probably due to the fact that the rate of pore pressure buildup during the shaking phase is high and is not much influenced by the rate of dissipation (even for Case 3 in which three fluid exit paths were available within the clay cap).
2. It is clear from the above study that the potential influence of sand boils on lateral site deformation strongly depends on evolution and extent of developed sand boil activity.

Overall, the above discussion highlights the potential impact of sand boil activity on lateral accumulated deformations. Dramatic differences may occur when highly impervious interfaces are present. In such cases, water/liquefied-sediment exit paths (already existing or caused by shaking/deformation) within these impervious interfaces can have significant engineering consequences on the extent of lateral deformation. Consequently, prediction of such lateral deformation may necessitate more accurate estimates of actual overall layer permeability (including presence/evolution of pore-pressure dissipation paths or sand boils). At present, this remains a difficult task, and reliance on earthquake reconnaissance data is essential for calibration of predictive tools.

6 SUMMARY AND CONCLUSIONS

The phenomenon and mechanism of sand boil formation were briefly discussed. Potential influence on liquefaction-induced lateral deformation was conceptually studied using an effective-stress solid-fluid fully coupled Finite Element program. In the presence of highly impervious interfaces, the numerical simulation results suggest that occurrence of sand boils might significantly reduce liquefaction-induced lateral deformations. At present, assessments of such sand boil activity are not feasible. Therefore, reliance on earthquake reconnaissance data is therefore necessary for calibration of related predictive tools.

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